SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2003
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections. Section A (Questions 1 and 2), Section B (Questions 3 and 4) and Section C (Questions 5 and 6).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 90 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *C. Kourtesis*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

SECTION A

Question 1 (15 marks) **Marks**

(a) Find $\sin 1\cdot 4$ **1**

(b) Express 450° in radians (in terms of π). **1**

(c) Differentiate

(i) $\sin 4x$ **1**

(ii) $(3x+1)^{20}$ **1**

(iii) $\cos(1-2x)$ **1**

(d) Find the gradient of the tangent to the curve $y = 4x^{3/2}$ at the point $(1,5)$. **2**

(e) If $y = x\tan x$, use the product rule to find $\frac{dy}{dx}$. **2**

(f) If $f(t) = \frac{1}{t}$, find the value of $f''(4)$. **2**

(g) Find

(i) $\int_1^9 \left(1+\sqrt{x}\right)dx$ **2**

(ii) $\int_0^{2\pi} \cos 2x \, dx$ **2**

Section A continued on the following page

Question 2 (16 marks) **Marks**

(a) (i) Sketch the graph of the function $y=(x+1)^2$. **1**

(ii) State the values of x for which the function is decreasing. **1**

(b) (i) Sketch the graph of $y=\sin x$ for $0 \le x \le 2\pi$. **1**

(ii) Find the area bounded by the curve $y=\sin x$ and the x axis for $0 \le x \le 2\pi$. **2**

(c) Find a primitive of $\dfrac{x+\sqrt{x}}{x}$. **2**

(d) In how many ways can:

(i) 6 different books be arranged on a shelf? **1**

(ii) 3 different paintings be hung in a row, from a collection of 8? **2**

(e) Find $\lim\limits_{x \to 0} \dfrac{\sin 7x}{5x}$ **2**

(f) The curve $y=5x^4-bx^2$ has a turning point at $x=1$. **2**

Find the value of b.

(g) If $g(x)=\sin^3 x$, find $g'\left(\dfrac{\pi}{4}\right)$ **2**

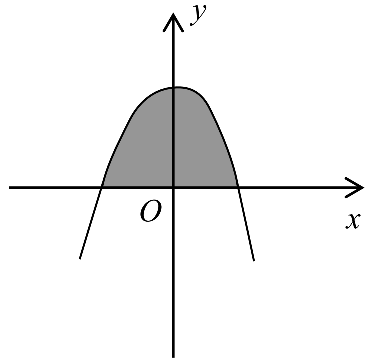
SECTION B (Start a NEW booklet)

Question 3 (14 marks) **Marks**

(a) For a certain curve $\frac{dy}{dx} = 2x - 5$ and $(2, -18)$ lies on the curve. **3**

Find the equation of the curve.

(b) **3**



The shaded region lying between the curve $y = 4 - x^2$ and the x - axis is rotated about the x – axis.

Find the volume of the resulting solid.

(c) Consider the curve $y = x^4 + 4x^3 - 16x + 1$

(i) Show that it has a minimum turning point at $x = 1$. **3**

(ii) Find the values of x for which the curve is concave up. **2**

(d) Prove by mathematical induction that, for all positive integers n, $3^{2n} - 1$ is divisible by 8. **3**

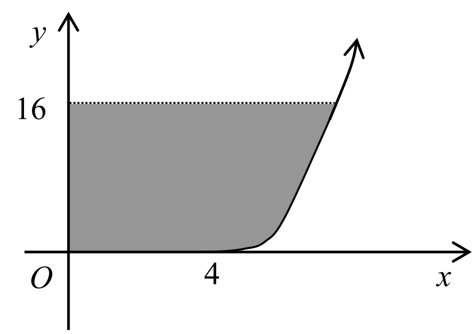
Section B continued on the following page

Question 4 (15 marks) **Marks**

(a) A committee of 3 men and 2 women is to be selected from a group of 10 men and 8 women. **2**

In how many ways can this be done?

(b)



The shaded area bounded by the curve $y = (x-4)^3$, the coordinate axes and the lines $y = 0$ and $y = 16$, is rotated about the y – axis.

(i) Show that the volume V of the solid formed is given by **2**

$$V = \pi \int_0^{16} \left(y^{2/3} + 8y^{1/3} + 16 \right) dy$$

(ii) Find the volume V in terms of π **3**

(c) Consider the function $f(x) = \dfrac{x}{x+1}$

(i) Find $f'(x)$. **2**

(ii) Show that there are NO stationary points. **1**

(iii) Show that the function is always increasing. **1**

(iv) Write down the equations of any asymptotes. **2**

(v) Sketch the graph of $y = f(x)$. **2**

SECTION C (Start a NEW booklet)

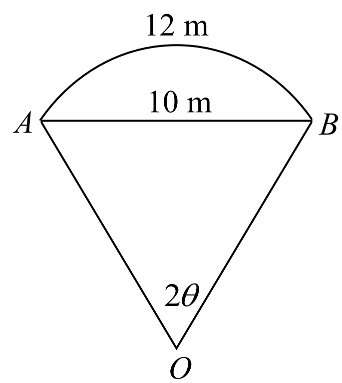
Question 5 (15 marks) **Marks**

(a) (i) Show that $\cos 2x = 1 - 2\sin^2 x$ **2**

(ii) Hence find $\int \sin^2 4x \, dx$ **3**

(b) Find the derivative of $\sin\left(\sqrt{\tan 2x}\right)$ **2**

(c)



O is the centre

A pipe which is 12 m long is bent into a circular arc which subtends an angle of 2θ at the centre of the circle.
The chord of the circle joining the ends of the arc is 10 m long.

(i) Show that $6\sin\theta - 5\theta = 0$ **4**

(ii) Show that $\theta_0 = 1^c$ is a good first approximation to the value of θ. **1**

(iii) Use one application of Newton's Method to find another approximation to θ. **3**

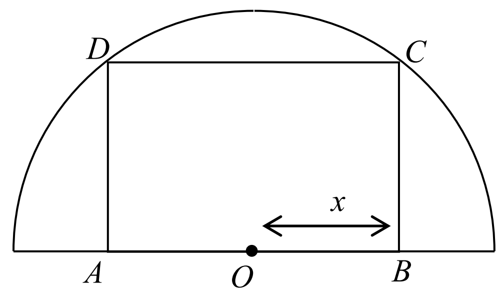
Section C is continued on the following page

SECTION C (continued)

Question 6 (15 marks) **Marks**

(a) Prove by mathematical induction that $4^n \geq n^2$ for all positive integers n **4**

(b)



A rectangle $ABCD$ is inscribed in a semi-circle of radius 10 units, centre O. Let $OB = x$.

(i) Show that the area A square units of the rectangle $ABCD$ is given by **2**

$$A = 2x\sqrt{100 - x^2}$$

(ii) Find the values of x for which $\frac{dA}{dx} = 0$ **3**

(iii) The rectangle $ABCD$ has a maximum area. **2**

Find the maximum area.

[You are **NOT** required to prove that the area is a maximum]

(c) Given the curve $y = x^{1/3} + \frac{1}{4}x^{4/3}$ find, giving reasons, any points of inflexion. **4**

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if}\ n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

~~May Assess 2003~~ 3 unit

Question 1

(a) $\sin 1.4 = 0.9854$ 1

(b) $450° = \frac{450 \times \pi}{180}^c$

$= \frac{5\pi}{2}^c$ 1

(c) (i) $\frac{d}{dx} \sin 4x = 4\cos 4x$ 1

(ii) $\frac{d}{dx}(3x+1)^{20} = 20(3x+1)^{19} \times 3$

$= 60(3x+1)^{19}$ 1

(iii) $\frac{d}{dx}\cos(1-2x) = -\sin(1-2x)(-2)$

$= 2\sin(1-2x)$ 1

(d) $y = 4x^{3/2}$ (1, 5)

$y' = \frac{3}{2} \times 4x^{1/2}$

$= 6x^{1/2}$

$y'(1) = 6$

$\therefore$ Gradient $= 6$ 2

(e) $y = x\tan x$

$\frac{dy}{dx} = \tan x + x\sec^2 x$ 2

(f) $f(t) = \frac{1}{t}$

$= t^{-1}$

$f'(t) = -t^{-2}$

$f''(t) = 2t^{-3}$

$= \frac{2}{t^3}$

$f''(4) = \frac{1}{32}$ 2

(g) (i) $\int_1^9 (1+\sqrt{x})\,dx$

$= \left[x + \frac{2x^{3/2}}{3}\right]_1^9$

$= (9+18) - (1+\frac{2}{3})$

$= 25\frac{1}{3}$

$= \frac{76}{3}$ 2

(ii) $\int_0^{2\pi} \cos 2x\,dx$

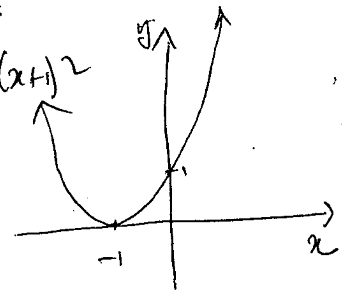
$= \frac{1}{2}\left[\sin 2x\right]_0^{2\pi}$

$= \frac{1}{2}[0 - 0]$

$= 0$ 2

Question 2

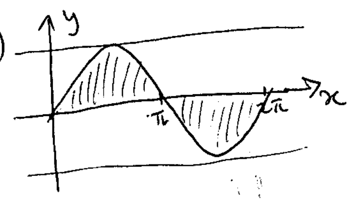
)(i) $y=(x+1)^2$



1

) Decreasing $= \{x : x \le -1\}$ 1

)(i)



1

$$\text{Area} = 2\int_0^{\pi} \sin x \, dx$$
$$= 2\left[-\cos x\right]_0^{\pi}$$
$$= 2\left[-(-1) - (1)\right]$$
$$= 4 \text{ sq. units}$$

2

) $$\int \frac{x+\sqrt{x}}{x}\,dx = \int \left(1 + x^{-\frac{1}{2}}\right) dx$$
$$= x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= x + 2\sqrt{x} + C$$

2

(d) (i) $6! = 720$ 1

(ii) $^8C_3 \times 3! = 56 \times 6$
$$= 336$$

1

(e) $$\lim_{x\to 0} \frac{\sin 7x}{5x} = \frac{7}{5} \lim_{x\to 0} \frac{\sin 7x}{7x}$$
$$= \frac{7}{5} \times 1$$
$$= \frac{7}{5}$$

2

(f) $y = 5x^4 - bx^2$
$y' = 20x^3 - 2bx$
T.P. at $x=1$ $\therefore y'(1) = 0$
$\therefore 0 = 20(1)^3 - 2b(1)$
$0 = 20 - 2b$
$2b = 20$
$b = 10$

2

(g) $g(x) = \sin^3 x$
$g'(x) = 3\sin^2 x \cos x$
$g'(\frac{\pi}{4}) = 3\sin^2 \frac{\pi}{4} \cos \frac{\pi}{4}$
$= 3\left(\frac{1}{\sqrt{2}}\right)^2 \cdot \left(\frac{1}{\sqrt{2}}\right)$
$= \frac{3}{2\sqrt{2}}$

2

Solutions Maths ext 1 YR12 2003 Task #2 May

③ (a) $y' = 2x - 5$

$y = \int (2x-5)\,dx$

$y = x^2 - 5x + C$

$(2,-18)$ $-18 = 4 - 10 + C$

$-18 = -6 + C$

$-18 + 6 = C$

$C = -12$

$\therefore y = x^2 - 5x - 12$ // ③

(c) $y = x^4 + 4x^3 - 16x + 1$

(i) $y' = 4x^3 + 12x^2 - 16$

$y'' = 12x^2 + 24x$

When $y' = 0$ $4x^3 + 12x^2 - 16 = 0$

if $x = 1$, $4 + 12 - 16 = 0$

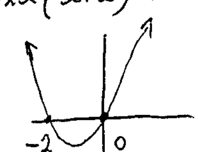
at $x = 1$, $y'' = 36 > 0$

at $x = 1$, min t. pt. ③

(ii) when $y'' > 0$

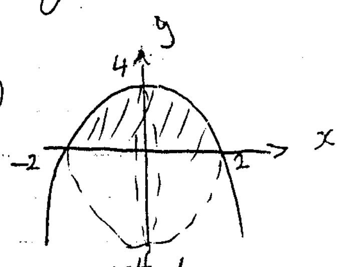
$12x^2 + 24x > 0$

$12x(x+2) > 0$



When $x < -2$, $x > 0$ ②

(b)



$y = 4 - x^2$

So $x^2 = 4 - y$.

$V = \pi \int_c^d y^2\,dy$

$= \pi \int_{-2}^{2} (4 - x^2)^2\,dx = \pi \int_{-2}^{2} (16 - 8x^2 + x^4)\,dx$

$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}$

$= \pi \left[32 - \frac{64}{3} + \frac{32}{5} - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right]$

③ $= \pi \left[34\frac{2}{15}\right]$ ③

$512\pi/15$ 3

3 (d) <u>step 1</u> let $n=1$, $8 / (3^2 - 1)$ is true.
when $n=1$, step 1 is true.

<u>step 2</u> If there exists a value $n=k$, where k is a positive integer then assume that $8 / (3^{2k} - 1)$ and we must prove that for $n=k+1$, $8 / (3^{2(k+1)} - 1)$ is true.

Now $3^{2k+2} - 1 = 3^{2k} \cdot 3^2 - 1$
$= 3^{2k} \times 9 - 1$
$= 9(3^{2k} - 1) + 8$

now $8 / 9(3^{2k} - 1)$ and $8/8$
$8 / (3^{2(k+1)} - 1)$

So $n=k+1$ is true.

<u>step 3</u> We have assumed it true for $n=1$, and $n=k$ and proved it true for $n=k+1$.
Hence statement is true for $n=2$, $n=3$ etc.
$\forall$ +ve integers n.

(3)

B ④ (a) ${}^{10}C_3 \times {}^{8}C_2 \qquad 120 \times 28 = 3360$ ②

(b) (i) $V = \pi \int_0^{16} x^2 \, dy.$ $\qquad y = (x-4)^3$

$= \pi \int_0^{16} (y^{\frac{1}{3}} + 4)^2 \, dy.$ $\qquad y^{\frac{1}{3}} = x - 4$, $x = y^{\frac{1}{3}} + 4$

$= \pi \int_0^{16} (y^{\frac{2}{3}} + 8y^{\frac{1}{3}} + 16) \, dy.$ ②

(ii) $V = \pi \left[\frac{3}{5} y^{\frac{5}{3}} + 6y^{\frac{4}{3}} + 16y \right]_0^{16}$

$V = \pi \left[\frac{3}{5} \times 16^{\frac{5}{3}} + 6 \times 16^{\frac{4}{3}} + 16 \times 16 \right]$

$= \pi \left[\frac{3}{5} \cdot (\sqrt[3]{16})^5 + 6 \times (\sqrt[3]{16})^4 + 256 \right]$

$= \pi \left[60.956 + 241.905 + 256 \right]$

$\doteqdot 558.861 \pi \, u^3.$ ③

(c) $f(x) = \frac{x}{x+1}$

(i) $f'(x) = \frac{(x+1) \times 1 - x \times 1}{(x+1)^2} = \frac{1}{(x+1)^2}.$ ②

(ii) $\frac{1}{(x+1)^2} = \frac{0}{1}$ x has no solution ①

(iii) except for $x = -1$, $y' = \frac{1}{(x+1)^2} > 0$ always ①

(iv) $f(x) = \frac{x}{x+1}$

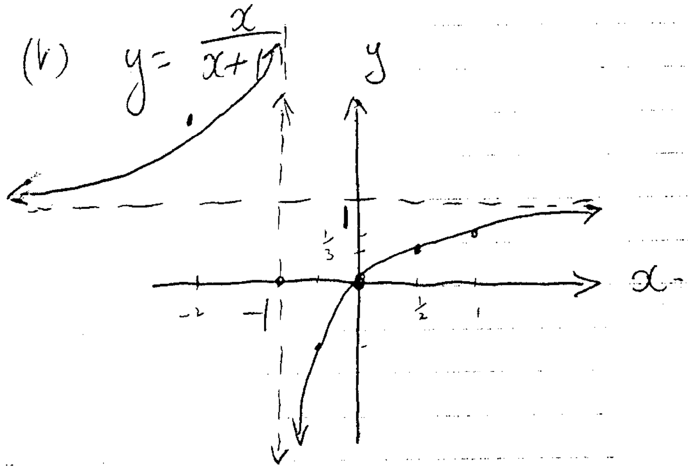
$x = -1$ is an assymptote.

(2)

$f(x) = \frac{x}{x(1+\frac{1}{x})} = \frac{1}{1+\frac{1}{x}}$

as $x \to \pm\infty$, $\frac{1}{x} \to 0$ $f(x) = 1$ is another assymptote

(v) $y = \frac{x}{x+1}$



(2)

15 marks. Section C — Solutions

Question (5)

(a) (i) $\cos 2x = \cos(x+x)$

$= \cos x\cos x - \sin x \sin x$

[2] $= 1 - \sin^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$.

$\therefore \sin^2 x = \frac{1}{2}(1-\cos 2x)$

(ii) $\int \sin^2 4x \, dx$

[3] $= \frac{1}{2}\int (1-\cos 8x)\,dx$

$= \frac{x}{2} - \frac{\sin 8x}{16} + c$

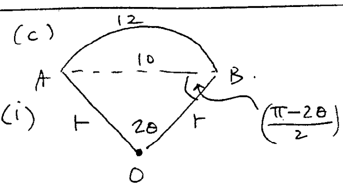
(b) $\frac{d}{dx}\sin(\sqrt{\tan 2x})$

$= \cos\sqrt{\tan 2x} \cdot \frac{1}{2}(\tan 2x)^{-\frac{1}{2}}$

[2] $\cdot 2\sec^2 2x$

$= \dfrac{\cos\sqrt{\tan 2x}\cdot \sec^2 2x}{\sqrt{\tan 2x}}$.

(c)



(i) Let r be the radius $[l = r\theta]$

[4] $12 = r(2\theta)$

$\therefore r = \frac{6}{\theta}$ — (1)

In $\triangle AOB$, apply sine rule

$\frac{10}{\sin 2\theta} = \frac{r}{\sin(\frac{\pi}{2}-\theta)}$

$\frac{10}{2\sin\theta\cos\theta} = \frac{r}{\cos\theta}$

$\therefore r = \frac{5}{\sin\theta}$ — (2)

Equate (1) and (2)

$\frac{6}{\theta} = \frac{5}{\sin\theta}$.

$\therefore 6\sin\theta = 5\theta$

i.e. $6\sin\theta - 5\theta = 0$

(ii) When $\theta = 1$

$f(\theta) = 6\sin(\theta) - 5\theta$

[1] $f(1) = 0.0488$

$f'(1) = -1.7582$.

$[f'(\theta) = 6\cos\theta - 5]$

[3] $\theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$

i.e. $\theta_1 = 1 - \frac{f(1)}{f'(1)}$.

$\therefore \theta_1 = 1 + \frac{0.04883}{1.758}$

$= 1.0278$.

Question (6)

(a) Let $S(n)$ be the statement that

$4^n \geq n^2$

For $n=1$, $4 > 1^2$

$\therefore S(1)$ is true.

Assume $S(k)$ is true

[4] i.e. $4^k > k^2$

Consider $n = k+1$

$4^{k+1} = 4 \cdot 4^k$

$> 4k^2$

$= k^2 + 3k^2$

Now, $3k^2 \geq 2k+1$

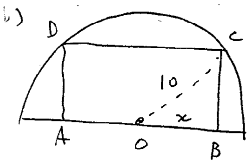
Whenever $k \geq 1$

$\therefore 4^{k+1} \geq k^2+2k+1$

i.e. $4^{k+1} \geq (k+1)^2$.

Since $S(n)$ is true for $n=1$ and true for $n=k+1$ when true for $n=k$ $(k \in \mathbb{Z}^+)$ $\therefore$ Statement is true

(b)



(i) $r = 10$, In $\triangle OBC$

[2] $10^2 = x^2 + BC^2$

$BC = \sqrt{100-x^2}$

Area of $ABCD = 2x\sqrt{100-x^2}$

(ii) $\frac{dA}{dx} = 2\sqrt{100-x^2} + 2x(100-x^2)^{-\frac{1}{2}} \times \frac{1}{2}\cdot(-2x)$.

[3] $\therefore \frac{dA}{dx} = 2\sqrt{100-x^2} - \frac{2x^2}{\sqrt{100-x^2}}$

$\frac{dA}{dx} = 0$ $\quad 100 - x^2 = x^2$

$\therefore 2x^2 - 100 = 0$.

$\therefore x^2 = 50$. $x = 5\sqrt{2}$

[2] $\therefore A_{max} = 2 \times 5\sqrt{2} \times 5\sqrt{2}$. $= 10\sqrt{2}(\sqrt{50})$

$=$

$= 100$

(c) $y = x^{\frac{1}{3}}(1+\frac{1}{4}x)$.

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}(1+\frac{x}{4}) + \frac{1}{4}x^{\frac{1}{3}}$

[4] $= \frac{1}{3}(x^{-2/3} + x^{\frac{1}{3}})$.

$\frac{d^2y}{dx^2} = \frac{1}{3}\left[-\frac{2}{3}x^{-5/3} + \frac{1}{3}x^{-\frac{2}{3}}\right]$

$= \frac{x^{-5/3}}{9}(x-2)$

$\frac{d^2y}{dx^2} = 0 \implies x = 2$.

x	1	2	3	−1
$f''(x)$	<0	0	>0	>0

When $x = 2$

$y = \sqrt[3]{2}\,(1+\frac{1}{2})$

$= \frac{3}{2}\sqrt[3]{2}$

$= 3 \cdot 2^{-2/3}$

$\therefore (2, 3\cdot 2^{-2/3})$ is a pt of